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REMARK ON LOCALIZATIONS OF NOETHERIAN RINGS WITH KRULL DIMENSION ONE

By

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Let R be a left noetherian ring with left Krull dimension α . For a left R -module M which has Krull dimension, we denote its Krull dimension by $K\text{-dim } M$ in this note. In the previous paper [6], we have shown that the family $\mathbf{F}_\beta(R) = \{ {}_R I \subseteq R \mid K\text{-dim } R/I \leq \beta \}$ is a left (Gabriel) topology on R for any ordinal $\beta < \alpha$. We are most interested in the case when R is (left and right) noetherian, $\alpha=1$ and $\beta=0$. Let R be such a ring and we denote $\mathbf{F}_0(R)$ by \mathbf{F} . Let A be the artinian radical of R . Then Lenagan [3] showed that R/A has a two-sided artinian, two-sided classical quotient ring $Q(R/A)$. In this note, we shall show that $R_{\mathbf{F}}$, the quotient ring of R with respect to \mathbf{F} , is isomorphic to $Q(R/A)$ as ring and we shall investigate a more precise structure of $R_{\mathbf{F}}$ under some additional assumptions.

In this note, a family of left ideals of R is said to be a topology if it is a Gabriel topology in the sense of Stenström's book [7]. So a perfect topology in this note is corresponding to a perfect Gabriel topology in [7]. Let \mathbf{G} be a left topology on R , and M a left R -module. A chain of submodules of M ;

$$M_0 \supseteq M_1 \supseteq \cdots \supseteq M_r$$

is called a \mathbf{G} -chain if each M_{i-1}/M_i is not a \mathbf{G} -torsion module. A \mathbf{G} -chain of M is said to be maximal if it has no proper refinement of \mathbf{G} -chain.

The following lemma can be proved easily.

LEMMA 1. *If ${}_R M$ has a finite maximal \mathbf{G} -chain of length r , then any \mathbf{G} -chain of M has a finite length s and $s \leq r$.*

Hence we can give a definition of \mathbf{G} -dimension of M , denoted by $\mathbf{G}\text{-dim } M$, as follows; if M has a finite maximal \mathbf{G} -chain of length r , define $\mathbf{G}\text{-dim } M = r$, and $\mathbf{G}\text{-dim } M = \infty$ otherwise.

COROLLARY 2. *For any short exact sequence of R -modules;*

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

we have $\mathbf{G}\text{-dim } M = \mathbf{G}\text{-dim } M' + \mathbf{G}\text{-dim } M''$.

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COROLLARY 3. Let $G \subseteq G'$ be left topologies on R , and M a left R -module. Then $G\text{-dim } M \geq G'\text{-dim } M$.

We apply Lenagan's results ([3, Theorem 3.6] and [2, Theorem 3.1]) in the following form.

THEOREM (Lenagan) Let R be a (left and right) noetherian ring with left Krull dimension one, and A its artinian radical. Denote R/A by \bar{R} , and $x+A$ by \bar{x} for $x \in R$. Let $S = \{s \in R \mid \bar{s} \text{ is a regular element in } \bar{R}\}$. Then the following statements hold.

- (1) $\Sigma(S) = \{Rs \mid s \in S\}$ is a cofinal family of F .
- (2) \bar{R} has a two-sided classical quotient ring $Q(\bar{R})$.

We should remark that Lenagan showed that $Q(\bar{R})$ is a (left and right) artinian ring. But in the assertion (2) we need only the existence of $Q(\bar{R})$ for our purpose.

In the following Lemmas 4, 5 and 6, R is assumed to be a left noetherian ring with left Krull dimension α .

LEMMA 4. (See [6, Theorem 3.1].) For any $\beta < \alpha$, $F_\beta = \{R I \subseteq R \mid K\text{-dim } R/I \leq \beta\}$ is a left topology on R .

LEMMA 5. Let t_{F_β} be the torsion radical corresponding to the topology F_β . Then $\text{rad}^\beta({}_R R) = t_{F_\beta}(R)$ where $\text{rad}^\beta({}_R R)$ is the largest left ideal of R whose Krull dimension is at most β . (Cf. [6])

PROOF. Clear by definitions.

LEMMA 6. For every left ideal I of R , $I \in F_\beta(R)$ if and only if $I + A/A \in F_\beta(R/A)$ where $A = t_{F_\beta}(R)$.

PROOF. Since $(R/A)/(I+A/A) \cong R/I+A$ as R/A -module and as R -module, $I \in F_\beta(R)$ implies that $K\text{-dim } R/I+A \leq K\text{-dim } R/I \leq \beta$. Thus $I+A/A \in F_\beta(R/A)$. Conversely assume that $I+A/A \in F_\beta(R/A)$. Then $K\text{-dim}_{R/A}(R/I+A) \leq \beta$. Since $I+A/I \cong A/A \cap I$, $K\text{-dim } I+A/I \leq K\text{-dim } A \leq \beta$. Thus $K\text{-dim } R/I \leq \beta$. Hence we have $I \in F_\beta(R)$.

In the sequel, R is assumed to be a left and right noetherian ring with left Krull dimension one. Denote $F_0(R)$ by F and $F_0(\bar{R})$ by F' respectively. Here $\bar{R} = R/A$ and $A = t_F(R)$.

LEMMA 7. $R_F \cong \bar{R}_F \cong Q(\bar{R})$ as ring.

PROOF. For any left ideal I of R , consider the following exact sequence:
 $0 \rightarrow I \cap A \rightarrow I \rightarrow I/I \cap A \rightarrow 0$. Since $I \cap A$ is an \mathbf{F} -torsion module and $\bar{R} = R/t_F(R)$ is \mathbf{F} -torsion-free, $\text{Hom}_R(I, \bar{R}) \cong \text{Hom}_R(I/I \cap A, \bar{R}) \cong \text{Hom}_{\bar{R}}(\bar{I}, \bar{R})$ where $\bar{I} = I + A/A$. Clearly the above isomorphisms are natural in I . Thus $R_F = \varinjlim_{I \in \mathbf{F}} \text{Hom}_R(I, \bar{R}) \cong \varinjlim_{I \in \mathbf{F}} \text{Hom}_{\bar{R}}(\bar{I}, \bar{R}) = \bar{R}_F$, by Lemma 6. It follows from Lenagan's theorem that $\bar{R}_F \cong Q(\bar{R})$ as ring. This completes the proof.

LEMMA 8. \mathbf{F} is a perfect topology.

PROOF. Let $S = \{s \in R \mid \bar{s} \text{ is a regular element in } \bar{R}\}$. Then by Lenagan's theorem, it is sufficient to prove that $bs = 0$ for $b \in R$ and $s \in S$ implies $ub = 0$ for some $u \in S$ (see [7, XI, Proposition 6.3]). Now we have then $\bar{b}\bar{s} = 0$ and hence $\bar{b} = 0$, that is, $b \in A$. Thus $Rb \cong R/\ell(b)$ is artinian and hence $\ell(b) \in \mathbf{F}$. Here $\ell(b)$ is the left annihilator ideal of b . It follows from Lenagan's theorem that $\ell(b) \cap S \neq \emptyset$. This shows that $\Sigma(S)$ is a cofinal family of \mathbf{F} .

Recall that R is said to satisfy the restricted minimum condition for left ideals if R/I is an artinian module for every dense left ideal I . (Cf. [6])

THEOREM 9. Let R be a noetherian QF-3 ring satisfying the restricted minimum condition for left ideals. Then R_F is a QF ring where $\mathbf{F} = \mathbf{F}_0$.

PROOF. By assumption, it follows from [6, Theorem 5.1] and [8, Proposition 1] that R has left Krull dimension at most one. Denote R_F by Q . Then by Lemma 8, Q_R is flat, and $Q \otimes_R N = 0$ if and only if N is an \mathbf{F} -torsion module for any left R -module N . Let M be any finitely generated left Q -module. Then it follows from the above facts that there exists a finitely generated, \mathbf{F} -torsion-free left R -module N such that $M \cong Q \otimes_R N$ as left Q -module. Since R satisfies the restricted minimum condition for left ideals, ${}_R N$ is \mathbf{D} -torsion-free where \mathbf{D} is the topology of dense left ideals. Since R is QF-3, ${}_R N$ is a finitely generated torsionless module. Thus ${}_R N$ can be embedded into a finitely generated free R -module because R is noetherian. Thus ${}_Q M$ can be embedded a finitely generated free Q -module. Since Q is a noetherian ring, it follows from the above facts that any proper descending chain of left ideals of Q is an \mathbf{F} -chain of ${}_R Q$. Since an R -module Q/\bar{R} is an \mathbf{F} -torsion module, we have $\mathbf{F}\text{-dim}_R Q = \mathbf{F}\text{-dim}_R \bar{R} = \mathbf{F}\text{-dim}_R R \leq \mathbf{D}\text{-dim}_R R < \infty$ by Corollary 3 and [8, Proposition 1]. This shows that ${}_Q Q$ has finite length. Therefore it follows from [4, Corollary 6] that Q is a QF ring.

THEOREM 10. *For a noetherian ring R , the following statements are equivalent.*

- (1) *R is a two-sided order in a QF ring and $K\text{-dim}_R R \leq 1$.*
- (2) *R can be decomposed into a ring direct sum, say $R = A \oplus B$, where A is a QF ring and B is a QF-3 ring satisfying the restricted minimum condition for left ideals and $\text{Soc}(B) = 0$.*

PROOF. Assume the statement (1). By [1, Theorem 10] we have a decomposition $R = A \oplus B$ where A is the artinian radical of R and $\text{Soc}(B) = 0$. By assumption, it is clear that A is a QF ring and B has a QF classical two-sided quotient ring $Q(B)$. Thus B is QF-3 (see [5, Theorem 1.5]). Let S be the set of all regular elements in B . Let $\Sigma(S) = \{Bs | s \in S\}$. Then $\Sigma(S) \subseteq F_0(B)$. Conversely assume $I \in \mathcal{D}(B)$ where $\mathcal{D}(B)$ is the topology of dense left ideal of B . Since ${}_{Q(B)}Q(B)$ is a cogenerator, we have $Q(B) = Q(B)I$ and hence $I \cap S \neq \emptyset$. Hence $F_0(B) = \mathcal{D}(B)$. This shows that B satisfies the restricted minimum condition for left ideals.

Conversely assume the statement (2). Then it is immediate from Lenagan's theorem, Theorem 9 and Lemma 7.

In the remainder of this note, we assume that R is a noetherian QF-3 ring satisfying the restricted minimum condition for left ideals. So R has left Krull dimension one. We denote the topology of dense left ideals by \mathcal{D} and F_0 by F . Let $S = R_F$ and $Q = R_{\mathcal{D}}$. Then we shall give a remark on the connection between two rings, S and Q . Now, we have a commutative diagram of canonical ring homomorphisms;

$$\begin{array}{ccc} R & \longrightarrow & Q \\ \phi \searrow & & \swarrow \psi \\ & S & \end{array}$$

because $\mathcal{D} \subseteq F$ by assumption.

Then we have

PROPOSITION 11. *Both ϕ and ψ are left flat epimorphisms. Moreover S is injective both as left R -module and as left Q -module.*

PROOF. By Lemma 8, ϕ is a left flat epimorphism and hence ψ is an epimorphism. By Theorem 9, S is an injective left S -module. So we see by adjointness that S is also injective as left R -module. Denote the artinian radical by A , and R/A by \bar{R} . Consider a canonical exact sequence;

$$0 \rightarrow R \rightarrow Q \rightarrow Q/R \rightarrow 0.$$

Since Q/R is a \mathbf{D} -torsion R -module, it is an \mathbf{F} -torsion module. On the other hand, ${}_R\bar{R}$ is \mathbf{F} -torsion-free and ${}_RS$ is an essential extension of ${}_R\bar{R}$. Thus ${}_RS$ is \mathbf{F} -torsion-free. Hence we have $\text{Hom}_R(Q/R, S) = 0$. For any left ideal I of Q and Q -homomorphism g of ${}_Q I$ into ${}_Q S$, there exists an R -homomorphism \bar{g} which makes the below diagram commutative;

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \longrightarrow & Q \\ & & g \downarrow & \nearrow \bar{g} & \\ & & S & & \end{array}$$

where j is an inclusion. Fix any element q_0 in Q . Define an R -homomorphism h of Q into S as follows;

$$qh = q(q_0\bar{g}) - (qq_0)\bar{g} \quad \text{for any } q \in Q.$$

It is clear that $Rh = 0$. We have the induced R -homomorphism \bar{h} such that the following diagram is commutative;

$$\begin{array}{ccc} Q & \xrightarrow{h} & S \\ \pi \searrow & & \nearrow \bar{h} \\ & Q/R & \end{array}$$

where π is the canonical map. By the above remark, $\bar{h} = 0$ and hence $h = 0$. This shows that \bar{g} is a Q -homomorphism and hence ${}_Q S$ is injective. It remains to show that S_Q is flat. Consider an exact sequence of left Q -modules:

$$0 \rightarrow X \rightarrow Y.$$

Since ${}_Q S$ is injective, we have the following exact sequence;

$$\text{Hom}_Q(Y, S) \rightarrow \text{Hom}_Q(X, S) \rightarrow 0.$$

Since ${}_S S$ is a cogenerator, it is immediate by adjointness that the following sequence is exact.

$$0 \rightarrow S \otimes_Q X \rightarrow S \otimes_Q Y.$$

Thus S_Q is flat. This completes the proof.

References

- [1] Ginn, S.M. and Moss, P.B.: A decomposition theorem for noetherian orders in artinian rings, Bull. London Math. Soc. 9 (1977) 177-181.
- [2] Lenagan, T.H.: Noetherian rings with Krull dimension one, J. London Math. Soc. (2) 15 (1977) 43-47.

- [3] Lenagan, T.H.: Artinian quotient rings of Macaulay rings, Lecture Notes in Math. 545, Springer, Berlin (1976).
- [4] Rutter, E.A.: A characterization of QF-3 rings, Pacific J. Math. 51 (1974) 533-536.
- [5] Sato, H.: On localizations of a 1-Gorenstein ring, Sci. Rep. Tokyo Kyoiku Daigaku A. 13 (1977) 188-193.
- [6] Sato, H.: On maximal quotient rings of QF-3 1-Gorenstein rings with zero socle, to appear in Osaka J. Math.
- [7] Stenström, B.: Rings of quotients, Grundlehren Wissenschaften 217, Springer, Berlin (1975).
- [8] Sumioka, T.: On non-singular QF-3' rings with injective dimension ≤ 1 , Osaka J. Math. 15 (1978) 1-11.

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